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Μαθημα 23ο

Διαφορ. Εξ.

Γραμ. ομογ. εξ. β' ταξης

$a_2 y'' + a_1 y' + a_0 y = 0$, $x_0 \in I$, $a_0, a_1, a_2 \in C(I)$

x_0 κανονικο ανωμαλο σημειο

($a_2(x_0) = 0$ ∨ $\frac{a_1}{a_2}$, $\frac{a_0}{a_2}$ μη αναλ. συν.)

Θεωρημα

Ας ειναι x_0 κανονικο ανωμαλο σημειο της (E_0^2). Ας ειναι $A_1 : D_1 \to \mathbb{R}$, $A_0 : D_0 \to \mathbb{R}$ με $x_0 \in D_0 \subseteq I$, $x_0 \in D_1 \subseteq I$ και

$A_1(x)\, a_2(x) = (x-x_0)\, a_1(x)$, $x \in D_1$

$A_0(x)\, a_2(x) = (x-x_0)^2 a_0(x)$, $x \in D_0$

και

$$A_1(x) = \sum_{n=0}^{\infty} p_n (x-x_0)^n, \quad |x-x_0| < R_1$$

$$A_0(x) = \sum_{n=0}^{\infty} q_n (x-x_0)^n, \quad |x-x_0| < R_2$$

Θετουμε $R = \min\{R_1, R_2\}$. Ας ειναι λ_1, λ_2 οι ριζες της (ενδεικτικης) εξισωσης

$\lambda^2 + (p_0 + 1)\lambda + q_0 = 0$ με $Re\lambda_1 \geq Re\lambda_2$

Τοτε

A) Μια λυση της (E_0^2): $y(x) = |x-x_0|^{\lambda_1} \sum_{n=0}^{\infty} c_n (x-x_0)^n$, $0 < |x-x_0| < R$

Μια αλλη λυση της (E_0^2) γραμ. ανεξ. της y_1:

i) $\lambda_1 - \lambda_2 \notin \mathbb{Z}$: $y_2(x) = |x-x_0|^{\lambda_2} \cdot \sum_{n=0}^{\infty} d_n (x-x_0)^n$, $d_0 \neq 1$

ii) $\lambda_1 = \lambda_2$: $y_2(x) = y_1(x) \log|x-x_0| + |x-x_0|^{\lambda_2} \sum_{n=0}^{\infty} d_n (x-x_0)^n$, $d_0 = 0$

iii) $\lambda_1 - \lambda_2 \in \mathbb{N}^*$: $y_2(x) = C\, y_1(x) \log|x-x_0| + |x-x_0|^{\lambda_2} \sum_{n=0}^{\infty} d_n (x-x_0)^n$, $d_0 = 1$

↳ αναλογα με την ασκηση προσδιοριζεται

Παράδειγμα 1, σελίδα 257

$$2x^2 y'' + (x - x^2) y' - y = 0 \quad , \quad x_0 = 0$$

Συντελεστές: $a_2(x) = 2x^2$

$a_1(x) = (x - x^2)$

$a_0(x) = -1$

$a_2(0) = 0 \rightarrow x_0 = 0$ ανώμαλο σημείο

- $A_1(x) = \dfrac{a_1(x)}{a_2(x)} (x - x_0) = \dfrac{(x - x^2)}{2x^2} x = \dfrac{-x^2(1-x)}{2x^2} = \dfrac{1}{2} + \dfrac{1}{2} x$

$R_1 = +\infty \quad , \quad p_0 = \dfrac{1}{2}$

- $A_0(x) = \dfrac{a_0(x)}{a_2(x)} (x - x_0)^2 = \dfrac{-1}{2x^2} x^2 = -\dfrac{1}{2} \quad , \quad R_2 = +\infty \quad , \quad q_0 = -\dfrac{1}{2}$

$R = \min\{R_1, R_2\} = +\infty$

$\lambda^2 + \left(\dfrac{1}{2} - 1\right)\lambda - \dfrac{1}{2} = 0$

$\lambda^2 - \dfrac{1}{2}\lambda - \dfrac{1}{2} = 0 \rightarrow \lambda_1 = 1 \quad , \quad \lambda_2 = -\dfrac{1}{2}$

Θα βρω τις λύσεις

$$y(x) = |x - x_0|^{\lambda_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n \quad , \quad c_0 = 1$$

$\rightarrow x > x_0$

$$y_1(x) = (x - 0)^1 \sum_{n=0}^{\infty} c_n (x - 0)^n$$

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+1} \quad , \quad x > 0 \quad , \quad c_0 = 1$$

στη 2η παραγώγιση χάνεται ο ένας όρος για $n=1$ όχι $n=2$

$$2x^2 \left(\sum_{n=1}^{\infty} c_n (n+1) n x^{n-1} \right) + (x - x^2) \sum_{n=0}^{\infty} c_n (n+1) x^n - \sum_{n=0}^{\infty} c_n x^{n+1} = 0 \quad , \quad x > 0$$

$$\sum_{n=1}^{\infty} c_n 2(n+1) n x^{n+1} + \sum_{n=0}^{\infty} c_n (n+1) x^{n+1} - \sum_{n=0}^{\infty} c_n (n+1) x^{n+2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0, \quad x>0$$

$$\downarrow$$

$$- \sum_{n=1}^{\infty} c_{n-1}\, n\, x^{n+1}$$

$$\cancel{c_0} x - \cancel{c_0} x + \sum_{n=1}^{\infty} \left(2 c_n (n+1) n + (n+1) c_n - n c_{n-1} - c_n\right) x^{n+1} = 0$$

↳ αναδρομικός τύπος

$$c_n (2n^2 + 2n + n + 1 - 1) - n c_{n-1} = 0, \quad n \geq 1$$

$$(2n^2 + 3n) c_n = n c_{n-1}, \quad n \geq 1$$

$$(2n+3) c_n = c_{n-1}, \quad n \geq 1$$

$$c_n = \frac{1}{2n+3} c_{n-1}, \quad n \geq 1 \qquad \text{δηλαδή} = 1$$

$$c_0 = 1$$

$$\left\{ \begin{aligned} c_1 &= \frac{1}{5} c_0 \\ c_2 &= \frac{1}{7} c_1 \\ &\vdots \\ c_n &= \frac{1}{2n+3} c_{n-1} \end{aligned} \right. \Rightarrow \quad c_n = \frac{1}{5 \cdot 7 \cdot \ldots \cdot (2n+3)}, \quad n \geq 1 \quad \Rightarrow$$

$$c_n = \frac{3}{3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n+1)}, \quad n \geq 1$$

$$y_2(x) = |x - x_0|^{\lambda_2} \cdot \sum_{n=0}^{\infty} d_n (x - x_0)^n, \quad d_0 = 1, \quad x_0 \neq 0$$

$$y_2(x) = |x|^{-1/2} \sum_{n=0}^{\infty} d_n x^n$$

$x>0$:

$$y_2(x)=\sum_{n=0}^{\infty} d_n x^{n-\frac{1}{2}} \quad , x>0 \quad , d_0=1$$

$$2x^2\left(\sum_{n=0}^{\infty} d_n(n-1/2)(n-3/2)\,x^{n-5/2}\right)+(x-x^2)\sum_{n=0}^{\infty} d_n(n-1/2)\,x^{n-3/2}-$$

$$-\sum_{n=0}^{\infty} d_n x^{n-\frac{1}{2}}=0$$

$$\sum_{n=0}^{\infty} 2d_n(n-1/2)(n-3/2)x^{n-\frac{1}{2}}+\sum_{n=0}^{\infty} d_n(n-1/2)x^{n-\frac{1}{2}}-\sum_{n=0}^{\infty} d_n(n-1/2)-\sum_{n=0}^{\infty} d_n x^{n-\frac{1}{2}}=0$$

$$\downarrow$$

$$\sum_{n=1}^{\infty} d_{n-1}(n-3/2)x^{n-\frac{1}{2}}$$

....

$$2n\,d_n-d_{n-1}=0 \quad , n=1,\ldots$$

$$d_n=\frac{1}{2n}d_{n-1} \quad , n=1,2,\ldots \qquad d_0=1$$

$n=1$: $d_1=\frac{1}{2}d_0=\frac{1}{2}$

$n=2$: $d_2=\frac{1}{2\cdot 2}d_1$ $\Rightarrow d_n=\frac{1}{2^n}\cdot\frac{1}{n!} \quad , n\geq 1 \quad , d_0=1$

$\vdots$

$d_n=\frac{1}{2n}\cdot d_{n-1}$

$x\neq 0$:

$$y_2(x)=|x|^{-\frac{1}{2}}\sum_{n=0}^{\infty} d_n x^n=|x|^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{1}{2^n}\cdot\frac{1}{n!}x^n=|x|^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(x/2)^n}{n!}=e^{x/2}$$

$$y_2(x)=|x|^{-1/2}e^{x/2} \quad , x\neq 0$$

παράδειγμα 2 (όταν έχω $\lambda_1=\lambda_2$)

$x^2y'' - (x^2+x)y' + y = 0$, $x_0 = 0$

Συντελεστές:

$a_2(x) = x^2$

$a_1(x) = -(x^2+x)$

$a_0(x) = 1$

$a_2(0) = 0$ άρα $x_0 = 0$ ανώμαλο σημείο

$A_1(x) = \frac{a_1(x)}{a_2(x)}(x-x_0) = \frac{-x(x+1)}{x^2}\cdot x = -x-1$, $p_0 = -1$, $R_1 = +\infty$

$A_0(x) = \frac{a_0(x)}{a_2(x)}(x-x_0)^2 = \frac{1}{x^2}\cdot x^2 = 1$, $q_0 = 1$, $R_2 = +\infty$

άρα $R = \min\{R_1, R_2\} = +\infty$

$\lambda^2 + (p_0-1)\lambda + q_0 = 0 \Rightarrow \lambda^2 - 2\lambda + 1 = 0$ ρίζες: $\lambda_1 = \lambda_2 = 1$ (διπλή)

$y_1(x) = |x|^1 \sum_{n=0}^{\infty} c_n x^n$, $c_0 = 1$

$x>0$: $y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+1}$, $c_0 = 1 \rightarrow c_n = \frac{1}{n} c_{n-1}$, $n = 1, 2, \dots$

$c_n = \frac{1}{n!}$, $n \geq 0$

Άρα $y_1(x) = x \sum_{n=0}^{\infty} \frac{1}{n!} x^n = xe^x$, $x>0 \Rightarrow \boxed{y_1(x) = |x|e^x \ , \ x \neq 0}$

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$y_2(x) = y_1(x)\log|x-x_0| + |x-x_0|^{r_1} \sum_{n=0}^{\infty} d_n (x-x_0)^n \quad (d_0=0)$

$\Rightarrow x>0:$

$y_2(x) = y_1(x)\log x + x\sum_{n=0}^{\infty} d_n x^n = y_1(x)\log x + \sum_{n=0}^{\infty} d_n x^{n+1} \quad (+1)$

$y_2'(x) = y_1'(x)\log x + y_1 \cdot \frac{1}{x} + \sum_{n=0}^{\infty} d_n (n+1) x^n \quad -(x^2+x)$

$y_2''(x) = y_1''\log x + 2y_1'(x)\frac{1}{x} + y_1(x)\left(-\frac{1}{x^2}\right) + \sum_{n=1}^{\infty} d_n(n+1)n x^{n-1} \quad x^2$

Προσθέτω

$0 = \log x \cancel{L(y_1)}^{0} + x^2\left(2y_1' \cdot \frac{1}{x} - \frac{1}{x^2}\cdot y_1 + \sum_{n=1}^{\infty} d_n(n+1)\cdot n x^{n+1}\right) - (x^2+x)$

$\left(y_1 \cdot \frac{1}{x} + \sum_{n=0}^{\infty}(n+1)d_n x^n\right) + \sum_{n=0}^{\infty} d_n x^{n+1}$

$\Rightarrow \quad 0 = x^2 2xe^x \frac{1}{x} - xe^x + \sum_{n=1}^{\infty} d_n(n+1)n x^{n+3} - x^2 e^x - \sum_{n=0}^{\infty}(n+1)d_n x^{n+2} - y -$

$- \sum_{n=0}^{\infty}(n+1)d_n x^{n+1} + \sum_{n=0}^{\infty} d_n x^{n+1} \quad \Rightarrow$

$0 = 2x^2\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) - x \sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=1}^{\infty}(n+1)n d_n x^{n+3} - x^2\sum_{n=0}^{\infty}\frac{x^n}{n!} -$

$- \sum_{n=0}^{\infty}(n+1)d_n x^{n+2} - \sum_{n=0}^{\infty}\frac{x^n}{n!} -$

$- \sum_{n=0}^{\infty}(n+1)d_n x^{n+1} + \sum_{n=0}^{\infty} d_n x^{n+1} \Rightarrow$

$\sum_{n=0}^{\infty}\frac{2}{n!}x^{n+2} - \sum_{n=0}^{\infty}\frac{1}{n!}x^n + \sum_{n=0}^{\infty}(n+1)n d_n x^{n+3} - \sum_{n=0}^{\infty}\frac{1}{n!}x^{n+2} - \sum_{n=0}^{\infty}\frac{1}{n!}x^n -$

$- \sum_{n=0}^{\infty}(n+1)d_n x^{n+1} + \sum_{n=0}^{\infty} d_n x^{n+1} = 0$

Παράδειγμα 3

$x^2y'' - xy' + 8(x^2-1)y = 0 \quad , x_0 = 0$

$a_2(x_0) = a_2(0) = 0$ άρα $x_0 = 0$ ανώμαλο σημείο

$A_1(x) = \frac{a_1}{a_2}(x)(x-x_0) = -\frac{x}{x^2} x = -1 = p_0 \quad , R_1 = +\infty$

$A_0(x) = \frac{a_0}{a_2}(x)(x-x_0)^2 = \frac{8(x^2-1)}{x^2} x^2 = -8+8x^2 \quad , q_0 = -8 \quad , R_2 = +\infty$

$R = \min\{R_1, R_2\} = +\infty$

$\lambda^2 + (p_0-1)\lambda + q_0 = 0 \Rightarrow \lambda^2 - 2\lambda + 8 = 0$

$\lambda_1 = 4 \quad , \lambda_2 = -2$

$y_1(x) = x^4 \sum_{n=0}^{\infty} c_n x^n \quad , c_0 = 1$

$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+4} \quad , c_0 = 1$

$\dots \quad y_1(x) = \dots$

$0 = \frac{1}{x}(-5d_1 + 8(-d_2+d_0)x + (-9d_3+8d_1)x^2 + 8(-d_4+d_2)x^3 + (-5d_5+3d_2)x^4 +$
$+ \sum_{n=0}^{\infty}(n(n+6)d_{n+6} + 8d_{n+4} + 2c(n+3)c_n)x^{n+5} = 0 \quad , n \geq 0$

$d_1 = 0 \quad , d_2 = d_0 = 0 \quad , d_3 = \frac{8}{4}d_1 = 0 \quad , d_4 = d_2 = 1 \quad , d_5 = \frac{3}{5}d_3 = 0$

άρα $n(n+6)d_{n+6} + 8d_{n+4} + 2c(n+3)c_n = 0$

$n = 0: \quad 8d_4 + 2c \cdot 3c_0 = 0 \Rightarrow c = -\frac{8d_4}{6c_0} = -\frac{4}{3}$

Εξίσωση Legendre

$$(1-x^2)y'' - 2xy' + m(m+1)y = 0 \quad , \; x_0 = 0$$

$$y(x) = \sum c_n x^n$$

- $m \in \mathbb{R}$

....

$$(n+1)(n+2)\,c_{n+2} + (m-n)(m+n+1)\,c_n = 0 \quad , \; n \geq 0$$

$$c_{n+2} = \frac{(m-n)(m+n+1)}{(n+1)(n+2)}\,c_n \quad , \; n \geq 0$$

αν $m = n \Rightarrow c_{m+2} = 0$ τότε $c_{m+1}, \ldots = 0$

- $m \in \mathbb{N}$, $n \in \mathbb{N}$, $m \neq n$

πολυωνυμική λύση , $P_m(x)$ (όπου βαθμός του ενός άλλος του άλλου)

$P_n(x)$

$$\underbrace{(1-x^2)P''_m - 2xP'_m}\; + m(m+1)P_m = 0$$

$$[(1-x^2)P'_m]' + m(m+1)P_m = 0 \quad \cdot P_n$$

$$[(1-x^2)P'_n]' + n(n+1)P_n = 0 \quad \cdot P_m \qquad \Rightarrow$$

$$P_n[(1-x^2)P'_m]' - P_m[(1-x^2)P'_n]' = [n(n+1) - m(m+1)]P_n P_m \quad , \; x \in \mathbb{R}$$

ολοκληρώνω στο $(-1,1)$

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